Uniqueness and Stability of Higher-Dimensional Black Holes[*]

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The present status of the uniqueness and stability issue of black holes in four and higher dimensions is overviewed with focus on the perturbative analysis of this issue for static black holes in higher dimensions as well as those in four dimensions with cosmological constant.

I.   INTRODUCTION

Does a gravitational collapse always lead to formation of a black hole? What kind of black holes can exist in nature? Can realistic black holes be well described by stationary solutions? These are among the most important problems in general relativity and have been studied for long time. Although we do not have a definite answer to the first question related to the cosmic censorship yet, we have rather well established answers to the second and third questions, which are summarised as the uniqueness and stability of asymptotically flat stationary regular black hole solutions.

In local astrophysical problems in the present universe, the assumption of the asymptotic flatness will be a good approximation. However, the success of the inflationary universe model implies that the cosmological constant cannot be neglected even locally in the early universe. Further, recent developments in unifying theories suggest the possibility that the universe has dimensions higher than four on microscopic scales[1, 2] and that mini black holes might be produced in elementary particle processes in colliders as well as in high energy cosmic shower events. Therefore, it is a quite important problem whether the uniqueness and stability of regular black hole solutions also hold in the non-vanishing cosmological constant case and/or in higher dimensions.

In this article, we briefly overview the present status of the investigations of this problem and discuss what kind of new information can be obtained by the linear perturbation theory.

II.   FOUR-DIMENSIONAL BLACK HOLES

A.   Uniqueness of black holes

1.   Asymptotically flat Einstein-Maxwell system

As mentioned in Introduction, we now have an almost complete list of regular stationary black hole solutions for the asymptotically flat Einstein-Maxwell system (for historical references, see the reviews Refs. [3, 4]).

What has played the most important role in this classification problem is the rigidity theorem, which asserts that for the Einstein-Maxwell system, an asymptotically flat stationary regular analytic black hole solution whose horizon is an analytic submanifold is axisymmetric if it is rotating, i.e., the Killing vector that represents the time translation at infinity is not parallel to the null generators on the horizon[3]. The first full proof of this theorem was given in the textbook by Hawking and Ellis[5] under the assumption that the horizon is connected and non-degenerate, i.e., the surface gravity of the horizon does not vanish. Incompleteness of their proof concerning the claims that the domain of outer communication is simply connected and the black hole surface has S^2 topology was later removed by Chruściel and Wald[6]. Further, the theorem was extended by Chruściel[3] to include the case in which the horizon has multiple components and/or is degenerate. At about the same time, it was proved by Rácz and Wald[7] that a non-rotating regular black hole solution is static if the horizon is non-degenerate.

Thus, the classification problem of asymptotically flat stationary (regular analytic) black hole solutions was reduced to that of static black holes and axisymmetric black holes. For neutral static black holes and charged static black holes with non-degenerate horizon, this classification problem is completely solved: the Schwarzschild and Reissner-Nordström solution are the only regular black hole solutions, as is well-known. In contrast, in the charged degenerate case, there exists in addition the family of the Majumdar-Papapetrou multiple black hole solutions, which was recently shown to exhaust all regular solutions with multiple black holes if the charges of all black holes have the same sign[8].

Next, in the rotating case, the Kerr-Newman solution is the only asymptotically flat regular solution, if the horizon is connected. Concerning multiple black hole solutions, Weinstein classified solutions that are regular except on the symmetry axis and found that for any positive integer N, there exist families of N black hole solutions, with parallel spins aligned on the symmetry axis, which has $3N - 1$ parameters for the vacuum case[9] and $4N - 1$ parameters for the electro-vacuum case[10]. At present, it is not known whether these families contains a completely regular solution without a strut singularity.

2. *Other asymptotically flat systems*

Uniqueness theorems are also proved for some asymptotically flat systems with other types of matter. For example, the only regular static black hole solutions with non-degenerate horizon are given by the Gibbons-Maeda solutions[11] for the Einstein-Maxwell-Dilaton system[12, 13, 14, 15, 16], and by the Reissner-Nordström solution for the Einstein-Maxwell-Dirac system. Further, it is also shown that the only regular stationary black hole solutions with non-degenerate horizon is the Kerr solution when the matter is described by a harmonic scalar[17]. Note that in this result, the rigidity theorem is assumed for the rotating case.

In the meantime, it is now known that the uniqueness theorem is not universal and does not hold for some systems. For example, for the Einstein-Yang-Mills system, it is shown that there exist a family of spherically symmetric regular soliton solutions and regular black hole solutions different from Schwarzschild solutions[18, 19, 20]. Although these additional solutions are shown to be unstable[21, 22, 23], the Einstein-Skyrme system allows of a spherically symmetric regular black hole solution with a Skyrme hair[24] that is stable[25, 26].

3. *Asymptotically dS and AdS systems*

The investigation of the classification of black hole solutions had been mainly concerned with the asymptotically flat case for a long time. This was because basic tools used to prove the uniqueness theorems for the asymptotically flat case do not work for systems with non-vanishing cosmological constant.

Recently, however, Anderson has worked out a new approach which leads to a new type of rigidity theorem and enables us to study the classification problem of static spacetimes with negative cosmological constant[27, 28]. Utilising this approach, Anderson, Chruściel and Delay showed that an asymptotically anti-de Sitter and static vacuum solution containing one non-degenerate regular black hole is unique and given by the Schwarzschild-AdS black hole solution if it has a C^2 conformal spatial completion[29]. Unfortunately, their method cannot be applied to the rotating case or to the case of positive cosmological constant. As far as the author knows, no uniqueness theorem with sufficient generality has been proved in the case of positive cosmological constant. Later, we will show that in a perturbative sense, the uniqueness of black hole solutions can be proved even for this positive cosmological constant case.

B. Stability of black holes

The uniqueness of black holes does not necessarily imply their stability. In particular, if the cosmic censorship hypothesis (CCH) does not hold, a regular stationary black hole solution will be unstable against a generic perturbation, because formation of naked singularity must be generic[30].

In reality, many of the standard black hole solutions are known to be perturbatively stable, supporting CCH. For example, both Schwarzschild black holes and non-degenerate Reissner-Nordström black holes are proved to be stable[31, 32, 33, 34, 35]. Recently, the Kottler black holes, i.e., the spherically symmetric black holes in spacetimes with non-vanishing cosmological constant, were also shown to be stable by the author and Ishibashi[36].

In contrast to static black holes, the stability of rotating black holes are established only for the Kerr black holes[37]. We do not discuss the stability of these rotating black holes in this article.

In the proofs of the stability of these black holes, a key role was played by the fact that the basic equations for perturbations can be reduced to a single second-order ODE (ordinary differential equation) of the Schrödinger type, which is often called a master equation. Now, we explain how such a master equation is derived and how the stability is proved in terms of it.

1. *Tensorial decomposition of perturbations*

In spacetimes of dimension $n + 2$, the linearisation of the Einstein equations yields the following equations for the metric perturbation $h_{\mu\nu} = \delta g_{\mu\nu}$:

$$
\begin{aligned}
(\triangle_L h)_{\mu\nu} - \nabla_\mu \nabla_\nu h + 2\nabla_{(\mu} \nabla^\alpha h_{\nu)\alpha} \\
+ (-\nabla^\alpha \nabla^\beta h_{\alpha\beta} + \triangle h + R^{\alpha\beta} h_{\alpha\beta}) g_{\mu\nu} \\
+ (2\Lambda - R) h_{\mu\nu} = 2\kappa^2 \delta T_{\mu\nu},
\end{aligned} \tag{1}
$$

where $\triangle_L$ is the Lichinerowicz operator defined by

$$
(\triangle_L h)_{\mu\nu} := -\nabla \cdot \nabla h_{\mu\nu} + 2R^\alpha_{(\mu} h_{\nu)\alpha} - 2R_{\mu\alpha\nu\beta} h^{\alpha\beta}. \tag{2}
$$

In order to analyse the behaviour of perturbations on the basis of these equations, the following two technical problems have to be resolved. Firstly, the perturbation variables $h_{\mu\nu}$ contain unphysical gauge degrees of freedom that should be eliminated. Second, these perturbation equations are a coupled system of a large number of equations and are quite hard to solve directly.

In the case of our interest in which the unperturbed background spacetime is spherically symmetric, these problems can be resolved with the help of the tensorial decomposition and the gauge-invariant formulation[38, 39, 40]. First, the metric of a spherically symmetric black hole solution to the vacuum or electro-vacuum system in $n + 2$ dimensions is written as

$$
ds^2 = g_{ab}(y)dy^a dy^b + r^2(y)d\Omega^2, \tag{3}
$$

where $g_{ab}dy^a dy^b$ is the metric of a two-dimensional orbit space $\mathcal{N}^2$, which can be written as $-f(r)dt^2 + f(r)^{-1}dr^2$, and $d\Omega^2 = \gamma_{ij}(z)dz^i dz^j$ is the metric of the unit sphere S^n. On this background, the basic perturbation variables

$h_{\mu\nu}$ can be grouped into three sets, h_{ab}, h_{ai} and h_{ij}, according to their tensorial transformation behaviour on S^n. Among these, h_{ai} can be further decomposed into the gradient part h_a and the divergence-free part $h_{ai}^{(1)}$ as

$$h_{ai} = \hat{D}_i h_a + h_{ai}^{(1)}; \quad \hat{D}^i h_{ai}^{(1)} = 0, \qquad (4)$$

where $\hat{D}_i$ is the covariant derivative with respective to the metric γ_{ij} on S^n. Similarly, h_{ij} can be decomposed into the trace part and trace-free part as

$$h_{ij} = h_L \gamma_{ij} + h_{Tij}; \quad h_{Tj}^j = 0, \qquad (5)$$

and the trace-free part h_{Tij} can be further decomposed into the scalar part $h_T^{(0)}$, the divergence-free vector part $h_{Ti}^{(1)}$ and the divergence-free and trace-free part $h_{Tij}^{(2)}$ as

$$h_{Tij} = \left(\hat{D}_i \hat{D}_j - \frac{1}{n}\gamma_{ij}\hat{\triangle} \right) h_T^{(0)} + 2\hat{D}_{(i} h_{Tj)}^{(1)} + h_{Tij}^{(2)};$$
$$\hat{D}^j h_{Tj}^{(1)} = 0, \ \hat{D}^j h_{Tij}^{(2)} = 0. \qquad (6)$$

By this decomposition, the linearised Einstein equations are decomposed into three decoupled sets of equations, each of which contains only variables belonging to one of the three sets of variables, the scalar-type variables $(h_{ab}, h_a, h_L, h_T^{(0)})$, vector-type variables $(h_{ai}^{(1)}, h_{Ti}^{(1)})$, and tensor-type variables $(h_{Tij}^{(2)})$. Further, since the covariant derivative $\hat{D}_i$ always appears in the combination $\hat{\triangle} := \hat{D}_i \hat{D}^i$ in the perturbation equations due to spherical symmetry, the expansion coefficients of these variables in terms of harmonic tensors on S^n, which are denoted as $(f_{ab}, f_a^{(0)}, H_L, H_T^{(0)})$, $(f_a^{(1)}, H_T^{(1)})$ and $H_T^{(2)}$ for the scalar-type, vector-type and tensor-type variables, respectively, are mutually coupled among only those corresponding to the same harmonic tensor (see Ref.[36, 41] for details of this harmonic expansion). Thus, the perturbation equations can be reduced to sets of equations on the two-dimensional orbit space $\mathcal{N}^2$ with small number of entries independent of the spacetime dimension. In addition, after this harmonic expansion, we can easily construct a basis for gauge-invariant variables by taking appropriate linear combinations of these two-dimensional variables and their derivatives with respect to the coordinates y^a of $\mathcal{N}^2$, and the linearised Einstein equations can be written as differential equations for these gauge-invariant variables[40]. Thus, we can essentially resolve the two technical problems mentioned at the beginning of this subsection.

2. Schwarzschild black hole

By applying the above arguments to four-dimensional spherically symmetric black holes, we obtain two decoupled sets of equations for the scalar-type and vector-type perturbations, which are often called the even modes (or

the polar perturbation) and the odd modes (or the axial perturbation), respectively. The tensor-type perturbation does not exist because there is no trace-free and divergence-free symmetric tensor on S^2. Since the background is static, by the Fourier transformation with respect to the time coordinate t,

$$h_{ab} = f_{ab}(r)e^{-i\omega t}\mathbb{Y}(\theta, \phi),$$
$$h_{ia}^{(1)} = f_a(r)e^{-i\omega t}\mathbb{Y}_i(\theta, \phi), \cdots, \qquad (7)$$

these equations can be further reduced to sets of ODEs for functions of r.

For the vector-type perturbation in the Schwarzschild background with $f(r) = 1 - 2M/r$, the corresponding set of equations can be easily reduced to a single second-order ODE of the Shcrödinger type,

$$-\frac{d^2\Phi}{dr_*^2} + V(r)\Phi = \omega^2\Phi; \quad r_* = \int^r \frac{dr}{f(r)}, \qquad (8)$$

as first shown by Regge and Wheeler[42]. It was shown by Zerilli later that a similar reduction is possible for the scalar-type perturbation of the Schwarzschild black hole as well[43]. Although these authors derived these master equations by the gauge-fixing method, the master variable Φ can be written in terms of gauge-invariant variables[44]. Thus, the linear stability of the Schwarzschild black hole is determined by the behaviour of the effective potential $V(r)$ for each mode.

The explicit expressions for the effective potentials are given by

$$V_V = \frac{f}{r^2}\left(m + 2 - \frac{6M}{r} \right), \qquad (9)$$

for the vector perturbation and

$$V_S = \frac{f}{r^2 H^2}\left[m^2(m+2) + \frac{6m^2 M}{r} + \frac{36mM^2}{r^2} + \frac{72M^3}{r^3} \right], \qquad (10)$$

for the scalar perturbation. Here, m is written in terms of the eigenvalue of the harmonic scalar $\triangle\mathbb{Y} = -l(l+1)\mathbb{Y}$ as $m = (l-1)(l+2)$ ($l = 2, 3, \cdots$), and $H = m + 6M/r$.

It is easy to see that the effective potentials V_V and V_S are positive outside the horizon ($r > 2M$). Because r_* runs in the range $(-\infty, +\infty)$, this implies that the master equation for Φ has no regular and normalizable solution for $\omega^2 < 0$. Hence, the 4D Schwarzschild black hole is perturbatively stable[31].

3. Reissner-Nordstöm black hole

For charged static black holes, metric perturbations couple perturbations of electromagnetic fields. However, by taking appropriate combinations of perturbation variables, the perturbation equations for each type can be reduced to two decoupled ODEs as

$$-\frac{d^2\Phi_\pm}{dr_*^2} + V_\pm(r)\Phi_\pm = \omega^2\Phi_\pm, \qquad (11)$$

where Φ_+ and Φ_- represent the electromagnetic and gravitational modes, respectively[35, 45, 46].

The effective potentials $V_\pm$ can be shown to be positive outside the horizon. Hence, the Reissner-Nordstöm black hole is perturbatively stable outside the (outer Killing) horizon, although it is unstable even against spherically symmetric perturbations inside the outer horizon (mass inflation[47]).

4. 4D Kottler black holes

Recently, Cardoso and Lemos found that the perturbation equations for 4D Schwarzschild-dS and -AdS black holes can be also reduced to a single master equation of the Schrödinger type[48](see also Ref.[44]).

First, the effective potential for the scalar perturbation is given by

$$V_S = \frac{f(r)}{r^2 H^2}\left[18x^2 f(r) + 27x^3 + 9(2 + 5l_2 + l_2^2)x^2 \right.$$
$$\left. +(l_2+1)^2(l_2+4)^2(l_2^2+5l_2+9)\right], \quad (12)$$

where $l_2 = l - 2 \geq 0$, $x = 2M/r$ and

$$f(r) = 1 - \lambda r^2 - \frac{2M}{r}. \quad (13)$$

Clearly, V_S is positive definite outside the horizon ($f(r) > 0$). Hence, there exists no normalizable mode with $\omega^2 < 0$.

Next, the effective potential for the vector perturbation is given by

$$V_V = \frac{f(r)}{r^2}\left[3f(r) + 3 + 3\lambda r^2 + l_2(l_2+5)\right]. \quad (14)$$

This effective potential is positive definite outside the horizon for $\lambda \geq 0$. However, for $\lambda < 0$ it becomes negative in a neighbourhood of the horizon if $|\lambda|$ is large. Nevertheless, we can show that there exists no normalizable $\omega^2 < 0$ mode for vector perturbations as well. One way to show this is to utilise the scalar-vector correspondence explained next.

5. The scalar-vector correspondence

Solutions Φ_1 and Φ_2 of the same frequency ω to two ODEs of the Schrödinger equation type are related by

$$\Phi_2(r) = p(r)\Phi_1(r) + q(r)\partial_r \Phi_1(r), \quad (15)$$

where $p(r)$ and $q(r)$ are functions of r that do not depend on ω, if and only if the effective potentials $V_1(r)$ and $V_2(r)$ for the two equations are expressed in terms of a single function $F(r)$ as

$$V_1, V_2 = \pm f\frac{dF}{dr} + F^2 + cF, \quad (16)$$

for some constant c. Such functions exist for the Regge-Wheeler and Zerilli potentials[35, 49].

This scalar-vector correspondence also holds for the Schwarzschild-dS and -AdS black holes[48]. Hence, we can conclude that there exists no normalizable vector perturbation with $\omega^2 < 0$ from the above result for scalar perturbations.

6. Perturbative uniqueness of 4D black holes

We can obtain some information about the uniqueness problem of black holes by the perturbation analysis. This is because there must exist a bounded regular solution with $\omega = 0$ to the perturbation equations if there were a continuous family of stationary regular black hole solutions that contains a spherically symmetric solution.

For example, by a method called S-deformation, which is explained later, we can show that there exists no bounded regular solution with $\omega^2 = 0$ outside the horizon for neutral as well as charged spherically symmetric static black holes, except for stationary vector perturbations with $l = 1$. These stationary solutions can be parametrised by three parameters that can be interpreted as the angular momentum of perturbations. Hence, we can conclude that 4D asymptotically dS or AdS static regular black hole solutions are unique in a perturbative sense under the conditions that the spacetime is static and contains a single black hole. This argument also suggests that the extensions of the Kerr solution to the non-vanishing cosmological constant case, which are described by the neutral Carter solutions with vanishing NUT parameter[50], are the only asymptotically dS or AdS stationary regular solutions with a connected horizon, in a vicinity of their static and spherically symmetric limit.

III. HIGHER-DIMENSIONAL BLACK HOLES

A. Generalised static solution

It is quite easy to find the higher-dimensional counterparts of 4D spherically symmetric black holes, such as the Tangherlini-Schwarzschild solution[51]. In reality, it is also possible to find slightly more general family of solutions for the Einstein-Maxwell system with cosmological constant Λ by requiring that the spacetime metric and the electromagnetic tensor $\mathcal{F}$ can be expressed in the form

$$ds^2 = g_{ab}(y)dy^a dy^b + r^2(y)d\sigma_n^2, \quad (17a)$$
$$\mathcal{F} = \frac{1}{2}E_0\epsilon_{ab}dy^a \wedge dy^b, \quad (17b)$$

where $g_{ab}dy^a dy^b$ is a metric of a two-dimensional space $\mathcal{N}^2$, and $\sigma_n^2 = \gamma_{ij}(z)dz^i dz^j$ is a metric of an Einstein
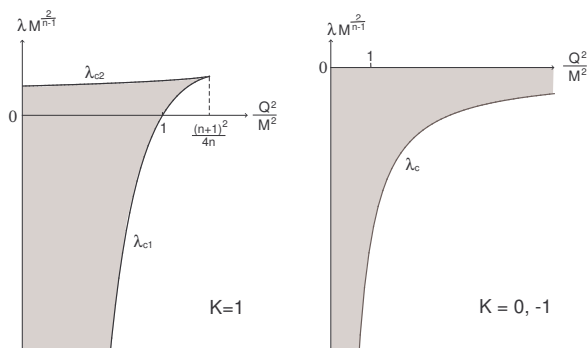
FIG. 1: The parameter ranges in which the spacetime contains a regular black hole (the shaded regions).

space $\mathcal{K}^n$ satisfying

$$\hat{R}_{ij} = (n-1)K\gamma_{ij}, \tag{18}$$

with $K = 0, \pm 1$. The Einstein equations and Maxwell equations determine the two-dimensional metric g_{ab} and the electric field E_0 as[36, 52]

$$g_{ab}dy^a dy^b = -f(r)dt^2 + \frac{dr^2}{f(r)}, \quad E_0 = \frac{q}{r^n}, \tag{19}$$

where

$$f(r) = K - \lambda r^2 - \frac{2M}{r^{n-1}} + \frac{Q^2}{r^{2n-2}}, \tag{20}$$

$$\lambda = \frac{2\Lambda}{n(n+1)}, \quad Q^2 = \frac{\kappa^2 q^2}{n(n-1)}. \tag{21}$$

In the special case in which $\mathcal{K}^n$ is the unit sphere S^n, the spacetime becomes static and spherically symmetric. In particular, for $K = 1, \Lambda = 0$ and $Q = 0$, this solution coincides with the Tangherlini-Schwarzschild solution, and for $K = 1$ and $\Lambda = 0$, it gives an higher-dimensional counter-part of the Reissner-Nordström solution. More generally, when $\mathcal{K}^n$ is a constant curvature space, the solution is invariant under $\mathrm{SO}(n+1)$, $\mathrm{ISO}(n)$ and $\mathrm{SO}(n,1)$ for $K = 1, 0, -1$, respectively.

In general, this solution have a naked singularity or does not have a horizon. As shown in Ref.[36], it gives a regular black hole spacetime only when the parameters M, Q and λ are in the special regions depicted in Fig.1. In this case, the Einstein space $\mathcal{K}^n$ describes a spatial section of the black hole horizon and at the same time the spatial infinity.

B. Uniqueness of black holes

In the static case with $\Lambda = 0$, the uniqueness of asymptotically flat regular black holes is now established in higher dimensions as well: all regular solutions are exhausted by the Tangherlini-Schwarzschild solution for the vacuum system[53, 54] and by the higher-dimensional

Reissner-Norsdtröm solution given by (17) and (19) with $K = 1$ and $\Lambda = 0$ in the non-degenerate case[55] and the higher-dimensional Majumdar-Papapetrou solutions in the degenerate case for the Einstein-Maxwell system[56]. Further, it is also proved that the Gibbons-Maeda solution and the Tangherlini-Schwarzschild solution are the only asymptotically flat regular black hole solutions for the Einstein-Maxwell-Dilaton system and for the Einstein-Harmonic-Scalar system, respectively[14, 16]. However, for $\Lambda \neq 0$, nothing is known about the uniqueness, although it was pointed out in Ref. [29] that the approach developed by Anderson may also apply to higher-dimensional cases with $\Lambda < 0$.

In the rotating case, the higher-dimensional counterpart of the Kerr solution is also known. It is given by the Myers-Perry solution[57], which has a spherical horizon and is asymptotically flat. In contrast to the static case, however, this is not the unique asymptotically flat rotating regular solutions in higher dimensions, although the uniqueness holds for supersymmetric black holes in the 5-dimensional $N = 1$ minimal supergravity model[58]. This is because the horizon topology need not be given by the sphere in higher-dimensions as discussed in Ref. [59]. In fact, Emparan and Reall[60] found an asymptotically flat and rotating regular black hole solution in five dimensions whose black hole surface is homeomorphic to $S^2 \times S^1$. Thus, at present, we have two different families of asymptotically flat rotating regular solutions with different horizon topologies. Since more complicated horizon topology is allowed in dimensions greater than 5, it is highly probable that other families of regular solutions exist. However, near the static and spherically symmetric limit, it is likely that a uniqueness theorem holds, as we will see later.

C. Stability of black holes

Recently, the author and Ishibashi have shown that for a static charge black hole in higher dimensions represented by (17) and (19), perturbation equations can be also reduced to decoupled 2nd-order master equations of the Schrödinger type, as in four dimensions[36, 44]. These master equations in higher dimensions, however, have some new features. First, the effective potential V is not positive definite in general. Hence, there may exist an unstable mode. Second, there exists no simple relation between vector and scalar perturbations like the scalar-vector correspondence in four dimensions, for $d = n + 2 > 4$. This implies that stabilities for scalar and vector perturbations should be studied separately. Third, there exist tensor perturbations for $d = n + 2 > 4$.

Now, let us see how these new features affect the stability of static black holes in higher dimensions.

1. Tensor perturbations

When $\mathcal{K}^n$ is a generic Einstein space, the $\mathcal{K}^n$-coordinates appear in perturbation equations only through the Lichnerowicz operator

$$\hat{\triangle}_L h_{ij} = -\hat{D} \cdot \hat{D} h_{ij} - 2\hat{R}_{ikjl} h^{kl} + 2(n-1)K h_{ij}. \quad (22)$$

Hence, by expanding tensor perturbations in terms of the eigentensors of $\hat{\triangle}_L$,

$$\hat{\triangle}_L \mathbb{T}_{ij} = \lambda_L \mathbb{T}_{ij}; \quad \mathbb{T}^i_i = 0, \quad \hat{D}^j \mathbb{T}_{ij} = 0, \quad (23)$$

we obtain a single decoupled equation for $H_T^{(2)}$, which can be easily put into the form (8).

In the special case in which $\mathcal{K}^n$ is a constant curvature space, $\hat{\triangle}_L$ is expressed as $\hat{\triangle}_L = -\hat{D} \cdot \hat{D} + 2nK$. Hence, the expansion by $\mathbb{T}_{ij}$ is identical to harmonic expansion, and the eigenvalue λ_T is related to the eigenvalue k_T^2 with respect to $-\hat{D} \cdot \hat{D}$ by $\lambda_L = k_T^2 + 2nK$. Note that k_T^2 is always non-negative and takes the discrete value $k_T^2 = l(l+n-1) - 2$ $(l = 2, 3, \cdots)$ for $\mathcal{K}^n = S^n$ $(K = 1)$.

For neutral black holes, the effective potential for tensor perturbations reads

$$V_T = \frac{f}{r^2} \left[\lambda_L - (3n-2)K + \frac{n(n+2)}{4} f + \frac{n(n+1)M}{r^{n-1}} \right]. \quad (24)$$

When $\mathcal{K}^n$ is maximally symmetric, $V_T > 0$ outside the horizon since $\lambda_L - (3n-2)K = k_T^2 - (n-2)K \geq 0$. Hence, the black hole is stable for a tensor perturbation. In contrast, when $\mathcal{K}^n$ is a generic Einstein space, no general lower bound for λ_L is known, and generalised static black holes can be unstable against a tensor perturbation[41].

Next for charged black holes, since EM field perturbations are not coupled to tensor perturbations of the metric, the master equation is again given by a single equation. The effective potential is given by

$$V_T = \frac{f}{r^2} \Big[\lambda_L - (3n-2)K + \frac{n(n+2)f}{4} + \frac{n(n+1)M}{r^{n-1}}$$
$$ - \frac{n^2 Q^2}{r^{2n-2}} \Big]. \quad (25)$$

We can show that this potential is positive definite outside the horizon if $\lambda_L - 2(n-1)K = k_T^2 + 2K \geq 0$. Hence, when $\mathcal{K}^n$ is maximally symmetric, the black hole is stable against tensor perturbations for $K \geq 0$. However, for $K = -1$ $(\lambda < 0)$, the potential becomes negative near the horizon if λ is close to the lower limit λ_{c-} in Fig.1. Hence, the black hole might be unstable in this case even when $\mathcal{K}^n$ is maximally symmetric.

2. S-deformation

For scalar and vector perturbations, we can expand perturbation variables by scalar harmonics $\mathbb{S}$ and vector harmonics $\mathbb{V}_i$ on $\mathcal{K}^n$ satisfying $\hat{\triangle}\mathbb{S} = -k^2\mathbb{S}$ and
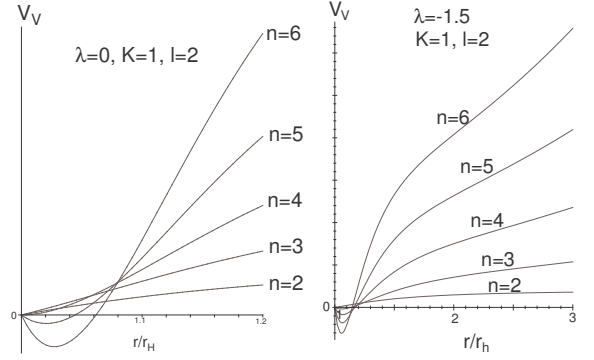

FIG. 2: Examples of $V_V = V_{V-}$ for $Q = 0$.

$\hat{D} \cdot \hat{D}\mathbb{V}_i = -k_V^2 \mathbb{V}_i$. For each harmonic, there appear two modes of perturbations, an electromagnetic mode Φ_+ and a gravitational mode Φ_-. They obey two decoupled equations of the form (11), as in the case of 4D Reissner-Nordström black hole. However, $V_\pm$ is not positive definite when n is large, even for spherically symmetric black hole. In order to study the stability for such a system, we utilised the following method that we call *the S-deformation method*.

Let I be the range of r_* corresponding to the regular region outside of the horizon, $-\infty < r_* < r_{*\infty}$. Here, $r_{*\infty}$ is $+\infty$ for $\lambda \geq 0$, but is finite for $\lambda < 0$. Then, in the space $C_0^\infty(I)$ of smooth functions with compact support, the operator

$$A := -\frac{d^2}{dr_*^2} + V(r) \quad (26)$$

is symmetric. We assume that A is extended to a self-adjoint operator in $L^2(I)$ by the Friedrichs extension. Then, the lower bound for the spectrum of A coincides with the lower bound of $\omega^2 = (\Phi, A\Phi)/(\Phi, \Phi)$ $(\Phi \in C_0^\infty(I))$. Here, for any regular function $S(r)$ on I, we have

$$(\Phi, A\Phi) = \int_I dr_* \left[\left| \left(\frac{d}{dr_*} + S(r) \right) \Phi \right|^2 + \tilde{V}|\Phi|^2 \right], \quad (27)$$

where

$$\tilde{V} = V + f\frac{dS}{dr} - S^2. \quad (28)$$

Hence, if we can show that $\tilde{V}$ is non-negative for an appropriate S, we can conclude that $\omega^2 \geq 0$, i.e., the stability of the system.
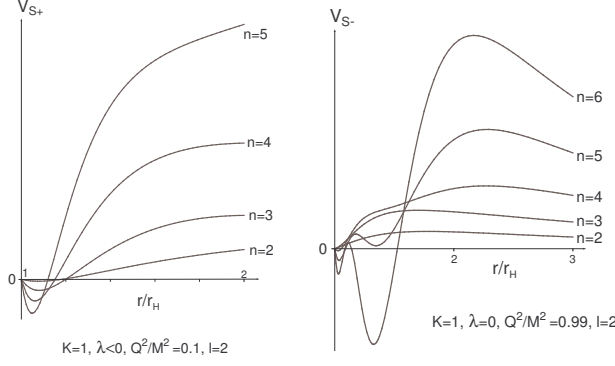
FIG. 3: Examples of $V_{S\pm}$

3. Vector perturbations

The effective potentials for vector perturbations are given by

$$V_{V\pm} = \frac{f}{r^2}\Big[k_V^2 + \frac{(n^2 - 2n + 4)K}{4} - \frac{n(n-2)}{4}\lambda r^2$$
$$+ \frac{n(5n-2)Q^2}{4r^{2n-2}} + \frac{-(n^2+2)M \pm 2\Delta}{2r^{n-1}}\Big], \quad (29)$$

where

$$\Delta^2 = (n^2-1)^2 M^2 + 2n(n-1)[k_V^2 - (n-1)K]Q^2. \quad (30)$$

These potentials are not positive definite when n is large as illustrated in Fig.2.

Nevertheless, we can show that there exists no unstable mode for most cases, by using the S-deformation method. In fact, for $S = nf/2r$, the S-deformation yields

$$\tilde{V}_{V\pm} = \frac{f}{r^2}\left[m_V + \frac{(n^2-1)M \pm \Delta}{r^{n-1}}\right], \quad (31)$$

where $m_V = k_V^2 - (n-1)K$. Since $m_V \geq 0$, $\tilde{V}_+$ is positive definite. Hence, the EM mode of a vector perturbation is always stable. Further, for $K \geq 0$, we can also show that $\tilde{V}_-$ is positive definite. Hence, we can conclude that black holes with $K \geq 0$ are stable against vector perturbations. In contrast, for $K = -1$ ($\lambda < 0$), $\tilde{V}_-$ becomes negative near the horizon when λ is close to the critical value λ_{c-}. In this case, we cannot prove the stability by this method.

4. Scalar perturbations

The explicit expressions of the effective potentials $V_{S\pm}$ for scalar perturbations are quite long. So, we only give the graphs of them for some special parameter values in Fig. 3. As these figures show, either of V_{S+} and V_{S-} is not positive definite in general.

However, by the S-deformation method again, we can show that the black hole is always stable for the EM mode perturbation[36]. We can also show that asymptotically

		Tensor		Vector		Scalar	
		$Q=0$	$Q\neq 0$	$Q=0$	$Q\neq 0$	$Q=0$	$Q\neq 0$
$K=1$	$\lambda=0$	OK	OK	OK	OK	OK	$d=4,5$ OK $d\geq 6$?
	$\lambda>0$	OK	OK	OK	OK	$d\leq 6$ OK $d\geq 7$?	$d=4,5$ OK $d\geq 6$?
	$\lambda<0$	OK	OK	OK	OK	$d=4$ OK $d\geq 5$?	$d=4$ OK $d\geq 5$?
$K=0$	$\lambda<0$	OK	OK	OK	OK	$d=4$ OK $d\geq 5$?	$d=4$ OK $d\geq 5$?
$K=-1$	$\lambda<0$	OK	?	OK	?	$d=4$ OK $d\geq 5$?	$d=4$ OK $d\geq 5$?

TABLE I: Stability of generalised static black holes

flat neutral black holes in arbitrary dimensions[61] and asymptotically dS and AdS neutral black holes in four dimensions are stable[44]. However, the stability of charged black holes has been proved only for special cases. The present status of the perturbative analysis of generalised static black holes is summarised in Table I.

D. Perturbative uniqueness

As explained in §II B 6, we can study the perturbative uniqueness of black holes near the static limit by looking for bounded stationary solutions to the perturbation equations. Actually, we can establish a kind of perturbative uniqueness theorem by this approach. For the limitation of space, we only give an outline of the argument here. The detail will be given in another paper[62].

First of all, note that there exist such solutions with $l = 1$ for the vector perturbation equation in the spherically symmetric case ($K = 1$) as shown in Ref.[61]. To be precise, vector harmonics on S^n are in one-to-one correspondence to Killing vectors of S^n, and for each harmonic there exists one stationary solution to the vector perturbation equation. A Killing vector on S^n is further in one-to-one correspondence to an antisymmetric matrix of rank $n + 1$, and the conjugate class of this matrix is classified by its N eigenvalues, where $N = [(n + 1)/2]$ is the rank of $SO(n + 1)$. Hence, these stationary solutions exactly correspond to the angular momentum freedom of the Myers-Perry solution[57].

In the asymptotically flat case, we can show that there exists no other stationary solution for which $\delta g_{\mu\nu}$ are bounded everywhere[44]. Hence, the Myers-Perry solution is the unique regular solution near the static limit. In the asymptotically de Sitter case, we can obtain the same result for the tensor and vector perturbations by using the S-deformation technique. The perturbative uniqueness for the scalar perturbation can also be proved in arbitrary dimensions by applying a similar argument to a master equation that holds only for stationary perturbations. In contrast, the asymptotically anti-de Sitter case is more subtle. In this case, there exist infinitely many solutions with bounded $\delta g_{\mu\nu}$ in addition to the $l = 1$ vec-

tor perturbations, in accordance with the general result in the four dimensional case[29]. However, if we require the stronger but natural asymptotic condition that every component of the metric perturbation with respect to an background orthonormal basis vanishes at $r \sim \infty$, these extra solutions are excluded.

IV. CONCLUDING REMARKS

In this article, we have overviewed the present status of the uniqueness and stability issue of black holes in four and higher dimensions. We have seen that the uniqueness and stability are well established for asymptotically flat neutral and static black holes in arbitrary dimensions and for asymptotically flat stationary black holes in four dimensions, but the situation concerning the other cases is quite unsatisfactory. In particular, the classification and stability analysis of rotating black holes and asymptotically de Sitter or anti-de Sitter black holes in higher dimensions are important open problems, although the perturbative analysis can give some useful information as we have shown. In order to solve these problems in the nonperturbative framework, it will be necessary to find a higher-dimensional extension of the rigidity theorem.

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